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purpose to give a long list of bibliographical aids to the mathematical student. Such a list may be found in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 12 (1903), pp. 408-426. Our main object has been to convey an accurate idea in reference to the magnitude of the total mathematical literature and some of the aids to use this literature wisely in the better libraries.

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THREE THEOREMS ON THE TRISECTION OF AN ACUTE ANGLE.

By J. SAMSONOFF, New York City.

THEOREM I. *The line DE (Fig. 1), which passes through the vertex A of one of the equal angles of an isosceles triangle ACB and intercepts on the line BC a part DC equal to the chord EC subtending the arc EC, which is drawn with radius AC from point A as a center, is the trisectorial line for the angle ABC.*

We have given the isosceles triangle ACB and the circumference FEC, which is drawn from the point A as a center and with AC as a radius, also the line DE which passes through A and intercepts on BC a part DC equal to the chord EC.

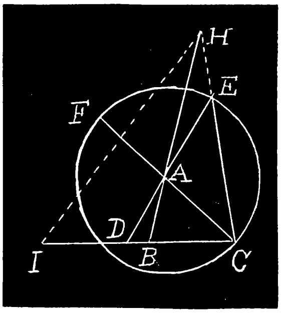

Fig. 1.

We are to prove that the line DE is the trisectorial line for $\angle ABC$, that is, $\angle DAD = \frac{\angle ABC}{3}$.

PROOF. Prolong the line CE until it intersects the line BA at the point H, and from H draw a parallel to ED, which will intersect the line DC at the point I.

Now, $\angle CAB$, the exterior of triangle HCA, is equal to $\angle ACH + \angle CHA$; $\angle ABC$, the exterior angle of triangle IHB, is equal to $\angle BIH + \angle IHB$; but triangle EDC is isosceles (by hypothesis), triangle HCI is isosceles (by construction, as HI is parallel to ED); and triangle CAE is isosceles (because $AC = AE$).

Therefore, $\angle ACH = \angle BIH$, and $\angle IHB = \angle CHA$. But $\angle IHB = \angle HAE$ (lines IH and DE are parallel by construction).

Therefore triangle AEH is an isosceles triangle and $AE = EH$. Hence, $\angle ABC = \angle DAB + \angle ADB = 3(\angle DAB)$, or line DE is the trisectorial line for $\angle ABC$.

THEOREM II. *The bisector CF of the angle at the vertex of an isosceles triangle ACB (Fig. 2), prolonged to the intersection with the trisectorial line DE, forms an isosceles triangle FEC.*

We have given the line CF bisecting $\angle ACB$, the vertical angle of an isosceles triangle ACB, and the trisectorial line DE, which is drawn through the vertex A of one of the equal angles of the same isosceles triangle ACB.

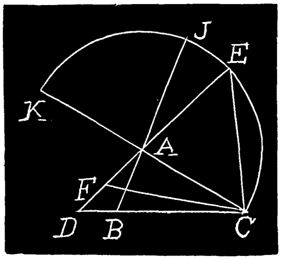

Fig. 2.

We are to prove that FEC is an isosceles triangle.

Proof. Prom the point A as a center and with AC as a radius describe a semicircle $KJEC$. As the line DE is the trisectorial line (by hypothesis), then $EC=DC$ (according to Theorem I); therefore DCE is an isosceles triangle.

But CAE is also an isosceles triangle ($AC=AE$).

Hence as DCE and CAE have a common angle AEC, we have $\angle ACE=\angle EDC$.

Then $\angle EFC=\angle EDC+\angle FCD$ ($\angle EFC$ is the exterior angle of triangle DFC); and $\angle ECF=\angle ECA+\angle ACF$. But $\angle ECA=\angle EDC$ (by construction), and $\angle ACF=\angle FCD$ (by hypothesis), hence the triangle FEC is isosceles.

Theorem III. *The chord DE (Fig. 3), which intersects the bisector AC of the angle at the vertex of an isosceles triangle DAB, forming the segment FE=BA, equal to the radius of the circumference, is the trisectorial line for the angle BDA.*

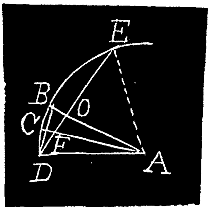

Fig. 3.

We have given the isosceles triangle DAB. From the point A as a center a semicircumference $DCBE$ is drawn with the radius AB. The line AC bisects $\angle BAD$, that is, $\angle BAC=\angle CAD$. The chord DE, which intersects AC at F, cuts off a segment $FE=AB$.

We are to prove that DE is the *trisectorial* line for $\angle BDA$, that is, $\angle BDE=\frac{\angle BDA}{3}$.

Proof. Join points A and E. Since $AE=AD$ (the radii of the same circumference), the triangle DAE is isosceles. Therefore, $\angle EDA=\angle DEA$. But triangle FEA is also isosceles ($FE=BA=EA$ by hypothesis). Hence $\angle EAF=\angle EFA$.

But $\angle EFA=\angle ADF+\angle FAD$, and $\angle EAF=\angle EAO+\angle OAF$. And as $\angle OAF=\angle FAD$ (by hypothesis), we have $\angle EAO=\angle FDA=\angle OEA$.

Hence, triangle AOE is also isosceles. Now, as the measure of $\angle OAE$ is arc BE, and the measure of $\angle BDE$ is $\frac{\text{arc } BE}{2}$, therefore $\angle OAE$ or $\angle ODA=2(\angle BDE)$. That is, DE is the trisectorial line for $\angle BDA$.

Problem. *To divide an acute angle into three egual parts by means of a graduated ruler and a compass.*

Solution. Let $\angle BAC$, an acute angle (Fig. 4), be divided by line AD into three equal parts. Take C as a center and with AC as a radius circumscribe a semicircle ABD, intersecting AB and AD at points B and D.

Connect points B and D with the center C. Let BC intersect AD at point O.

Triangle ACD is isosceles ($AC=CD$). Therefore $\angle DAC=\angle ADC=2(\angle BAD)$.

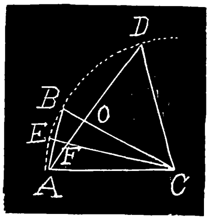

Fig. 4.

But triangle COD is also isosceles since the measure of $\angle BCD$ is arc BD; and the measure of $\angle BAD$ is one half of arc BD, making $BCD=2BAD=\angle DAC$. Therefore $\angle OCD=\angle ODC$.

Bisect $\angle BCA$, and let the bisector EC intersect AD at F. Now, triangle FDC is isosceles, for $DFC=\angle DAC+\angle FCA$; $\angle DCF=\angle DCO+\angle OCF$; $\angle DAC=\angle DCO$ as just proved, and $\angle OCF=\angle FCA$ by construction (we assumed EC is the bisector of $\angle BCA$).

Therefore $\angle DFC=\angle DCF$, and line $DF=DC$ equals the radius of the circle. From this analysis we come to the construction of the *trisectorial* line.

Assuming that the acute angle BAC (Fig. 5), is one of the equal angles of an isosceles triangle, we construct the isosceles triangle ACB, which will include the given angle BAC as an angle at the base of the isosceles triangle. Circumscribe circumference ABC', from point C as a center and with AC as a radius. Bisect $\angle BCA$.

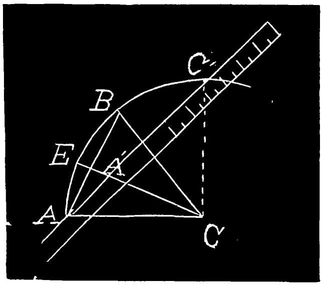

Fig. 5.

Then take the ruler and lay off on it a segment $A'C'$ equal to the radius AC. Bring the edge of the ruler to point A and draw line AC' in such a way that A' shall be on the bisector EC and C' on the circumference. The line AC' is the *trisectorial* line for $\angle BAC$.

Proof. Join C' and C; triangle $A'C'C$ is isosceles, because $A'C'=C'C$ (by construction); triangle ACC' is also isosceles ($AC=CC'$).

Now, $\angle C'A'C=\angle C'AC+\angle A'CA$; $\angle C'CA'=\angle C'CB+\angle BCA'$. But $\angle BCA'=\angle A'CA$ (by construction). Hence $\angle C'CB=\angle C'AC$. But $\angle BCC'=2(\angle BAC')$ since the measure of $\angle BCC'$ is arc BC', and the measure of $\angle BAC'$ is one half of arc BC'.

Hence, $\angle C'AC$ is also equal to $2(\angle BAC')$, and therefore $\angle BAC=3(\angle BAC')$, which makes the line AC' the *trisectorial* line.